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LETTER TO THE EDITOR

A simple improvement of the Migdal–Kadanoff renormalisation group scheme

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Abstract. We construct a family of renormalisation group transformations R_α both for the Ising model on a triangular lattice and for the Ising model with checkerboard interactions on a square lattice. For all parameter values α, R_α implies a lower bound to the free energy. The usual Migdal–Kadanoff scheme is recovered for $\alpha = 0$. However, the optimal value for the thermal scaling index y_t is obtained for $\alpha = \alpha^* \neq 0$ with $\partial y_t(\alpha)/\partial\alpha|_{\alpha^*} = 0$.

The Migdal–Kadanoff renormalisation group (MKRG) scheme (Migdal 1975, Kadanoff 1976) has been applied to a large variety of lattice systems (for a recent list of references see Burkhardt (1981)). In general, this simple scheme yields the correct phase diagram for all dimensions D. It can also be used to calculate thermodynamic functions which are quite accurate over the whole temperature regime (e.g. Lipowsky and Wagner 1981). However, the scaling indices are only reliable near the lower critical dimension of the phase transition considered. The standard example is the Ising model. For dimension $D = 1+\varepsilon$, one finds the thermal scaling index $y_t = 1/\varepsilon$ which is probably exact (Wallace and Zia 1979). For $D = 2$, the MKRG scheme yields $y_t = 0.747$, compared with the exact value $y_t = 1$. There have been several attempts to improve this value (Swendson and Zia 1979, Burkhardt 1979, Maritan 1980, Martinelli and Parisi 1981, Caracciolo 1981). In this Letter, we report on another attempt in this direction.

First, consider the two-dimensional Ising model on a triangular lattice with isotropic coupling constant K. In figure 1, we depict two hypertriangles of this lattice. In the usual MKRG scheme with rescaling factor $b = 2$, all interior bonds (broken bonds in figure 1) are moved onto the bonds forming the boundary of the hypertriangles (full bonds in figure 1) (Berker *et al* 1978). After this bond moving step, one may sum over those spins indicated by white vertices in figure 1. This procedure yields the recursion relation

$$K' = \tfrac{1}{2}\ln[\cosh(4K)] \tag{1}$$

which implies the critical coupling constant $K^c = 0.305$ and the thermal scaling index $y_t = 0.747$. The bond moving step is necessary in order to avoid the generation of new types of interaction terms. To achieve this goal, however, it is not necessary to move all interior bonds. All that is required is that the interior bonds of every second hypertriangle are shifted. For instance, in figure 1, we move only the three interior bonds of the unshaded hypertriangle. In addition, we are free to rearrange the nine bonds of the shaded hypertriangle. We denote the three interior bonds by K_i and the six boundary bonds by K_a. We introduce a parameter α and put $K_i = 2\alpha K$ and $K_a = (2-\alpha)K$ such

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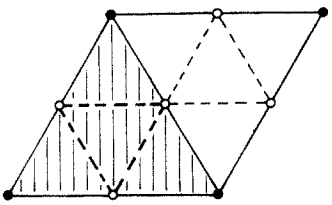

Figure 1. Two hypertriangles of the Ising model on a triangular lattice. Spins are indicated by black and white vertices.

that $3K_i + 6K_a = 9K$. $\alpha = 0$ corresponds to the usual MKRG scheme. Summing again over the spins indicated by white vertices in figure 1, we arrive at the recursion relation

$$K' = \tfrac{1}{4} \ln \left(\cosh^2(2K_a) + 3 \sinh^2(2K_a) \frac{e^{4K_i} - 1}{e^{4K_i} + 3} \right). \tag{2}$$

For large K, this recursion relation reduces to $K' = (2 - \alpha)K$. Thus for $\alpha \geqslant 1$, all RG trajectories approach the high-temperature fixed point. Therefore, we will restrict the parameter range to $\alpha < 1$. In order to see if the above RG transformation (2) with $0 \leqslant \alpha < 1$ is a good approximation at low temperatures ($K \to \infty$) we consider the recursion relation for the approximate bulk free energy f_B. This recursion relation has the generic form

$$f_B(K) = \tfrac{1}{4}[f_B(K') + \delta f_B(K; \alpha)] \tag{3}$$

for $D = 2$ and rescaling factor $b = 2$. In this case

$$\delta f_B(K; \alpha) = \ln [2\cosh(2K_a)(e^{3K_i} + 3e^{-K_i})] + K' \tag{4a}$$

with K' given by (2). For low temperatures we obtain

$$\delta f_B(K; \alpha) \xrightarrow[K \to \infty]{} 3(2 + \alpha)K. \tag{4b}$$

This asymptotic form for δf_B implies that the approximate bulk free energy has the exact asymptotic behaviour $f_B(K) \xrightarrow[K \to \infty]{} 3K$ irrespective of our choice for α.

For every α with $0 \leqslant \alpha < 1$, the recursion relation (2) has a critical fixed point with a corresponding scaling index $y_t(\alpha)$. If we had a family of exact RG transformations we would have $\partial y_t(\alpha)/\partial \alpha \equiv 0$ since y_t is a universal quantity. In our approximate scheme, this does not hold in general. However, it turns out that there is a unique value $\alpha = \alpha^*$ with $\partial y_t(\alpha)/\partial \alpha|_{\alpha^*} = 0$. For the triangular case, $\alpha^* = 0.40$ and $y_t(\alpha^*) = 0.849$. The corresponding critical coupling constant is $K^c(\alpha^*) = 0.301$ compared with the exact value $K^c = 0.274$.

Note that the RG transformation described above implies a lower bound to the free energy for all α (Kadanoff 1976, 1977). Thus, we may apply the Kadanoff criterion (Kadanoff *et al* 1976) to our bond moving scheme. According to this criterion, the optimal choice α^{**} for the parameter α is obtained from

$$\frac{d}{d\alpha} f_B(K^c(\alpha); \alpha)\Big|_{\alpha^{**}} = \frac{1}{3} \frac{d}{d\alpha} \delta f_B(K^c(\alpha); \alpha)\Big|_{\alpha^{**}} = 0. \tag{5}$$

For the transformation given by (2) and (4a), we find the unique value $\alpha^{**} = 0.29$ which

is rather different from $\alpha^* = 0.40$ obtained by our criterion $\partial y_t(\alpha)/\partial\alpha|_{\alpha^*} = 0$. The corresponding scaling index is $y_t(\alpha^{**}) = 0.840$.

Recently, Martinelli and Parisi (1981) have constructed another improvement of the MKRG scheme. Instead of moving the interior bonds K (broken bonds in figure 1), they move only a fraction $(1-\varepsilon)K$ of these bonds. $\varepsilon = 0$ corresponds to the usual MKRG scheme while $\varepsilon = 1$ means no (bond moving) approximation at all. Thus, by expanding in ε around $\varepsilon = 0$ they hope to get closer and closer to the exact quantities. When applied to the Ising model on a triangular lattice this procedure yields $y_t = 0.861$ up to $O(\varepsilon^2)$ if one naively puts $\varepsilon = 1$ (Caracciolo 1981). If the series in ε is recorded in such a way that the exact relation $\partial 2^{y_t}/\partial\varepsilon|_{\varepsilon=1} = 0$ holds up to $O(\varepsilon^2)$, one obtains $y_t = 0.859$. Thus our simple scheme with $\alpha = \alpha^*$ yields a scaling index comparable to the $O(\varepsilon^2)$ result of this asymptotic expansion. Of course, one may construct an ε expansion with the zeroth order given by our improved MKRG scheme. However, the algebra is quite involved since two different next-nearest-neighbour couplings and three different four-spin couplings have to be taken into account already in $O(\varepsilon)$.

Next, consider the two-dimensional Ising model on a square lattice. The usual MKRG scheme does not distinguish the square lattice from the triangular one. However, if we now try to improve the MKRG scheme in a similar fashion as we did for the triangular case, we generate both next-nearest-neighbour and four-spin couplings in addition to nearest-neighbour ones. One may move these new couplings in such a way that they do not affect the recursion relations for the nearest-neighbour coupling. Such a scheme has been considered by Maritan (1980) with rescaling factor $b = 3, 4$. Instead, we take the new couplings into account and consider an Ising model with checkerboard interactions as indicated in figure 2(a). In this figure, diagonal bonds represent next-nearest-neighbour couplings while circles represent four-spin couplings. The sub-Hamiltonian corresponding to every second square in figure 2(a) is given by

$$\begin{aligned}\mathcal{H}(\boldsymbol{K};\sigma_1\sigma_2\sigma_3\sigma_4) &= K_1(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)-\tfrac{1}{2}K_2[(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_4)^2] \\ &\quad +\tfrac{1}{4}K_3(\sigma_1-\sigma_3)^2(\sigma_2-\sigma_4)^2 \qquad (6)\end{aligned}$$

where $\sigma_1, \ldots, \sigma_4$ are spin variables and $\boldsymbol{K} := (K_1, K_2, K_3)$. The usual Ising model is recovered from (6) for $K_2 = K_3 = 0$. In addition, (6) is equivalent to a two-dimensional pure $\mathbb{Z}_2$-gauge model if we put $K_1 = 0$ and $K_2 = K_3$.

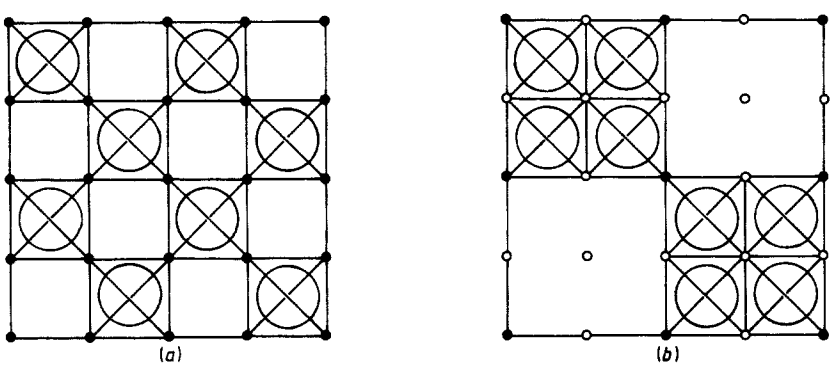

Figure 2. Ising model with checkerboard interactions on a square lattice (a) before and (b) after the bond moving step.

Starting with the model defined by (6), we move bonds as indicated in figures 2(a) and 2(b). As in the triangular case, we rearrange the nearest-neighbour couplings such that we obtain $K_a = (2-\alpha)K$ for the bonds on the boundaries of the hypersquares and $K_i = 2\alpha K$ for the interior bonds of the hypersquares. Summing over those spins indicated by white vertices in figure 2(b), we obtain a checkerboard Ising model with changed coupling constants $\boldsymbol{K}' = (K_1', K_2', K_3')$. The recursion relations for the coupling constants have the generic form

$$\boldsymbol{K}' = \boldsymbol{r}(\boldsymbol{K}; \alpha) \tag{7}$$

with $\boldsymbol{r} := (r_1, r_2, r_3)$, while the generic form for the recursion relation of the bulk free energy is still given by (3) with K replaced by $\boldsymbol{K}$. For $\alpha = 0$, we again obtain the usual MKRG scheme since $r_1(K_1, 0, 0; 0) = \frac{1}{2}\ln[\cosh(4K_1)]$. Thus, we recover the recursion relation (1) from (7) in this case. Although the RG transformation (7) is only approximate in general, it is exact for the special coupling constant subspace $\boldsymbol{K} =: (0, x, x)$ corresponding to the pure $\mathbb{Z}_2$-gauge model. In this case, we obtain from (7) the exact recursion relation

$$x' = \tfrac{1}{2}\ln\{\cosh[\ln(\cosh 2x)]\}.$$

For large coupling constant K_1, the recursion relations (7) reduce to

$$K_1' = (2-\alpha)K_1, \tag{8a}$$

$$K_2' = \tfrac{1}{4}\ln 2, \tag{8b}$$

$$K_3' = \begin{cases} -3\alpha K_1, & 0 \leqslant \alpha < \frac{1}{2}, \\ -(2-\alpha)K_1, & \frac{1}{2} \leqslant \alpha < 1, \end{cases} \tag{8c}$$

and

$$\delta f_B(\boldsymbol{K}; \alpha) = (4+2\alpha)K_1. \tag{8d}$$

(8a) indicates that we have to restrict the parameter α to the range $0 \leqslant \alpha < 1$ just as for the triangular case. 8(c) shows that the coupling constant K_3 becomes negative under renormalisation as soon as K_1 becomes large even if we start with an initial $K_3 \geqslant 0$. Thus, one may wonder if (8) describes the approach to the ferromagnetic low-temperature fixed point. However, the asymptotic form (8d) for δf_B implies that the approximate bulk free energy has the exact asymptotic behaviour $f_B(\boldsymbol{K}) \to 2K_1$ (irrespective of our choice for α). Thus, the RG transformation (7) indeed maps the model (6) onto a ferromagnetic ground state if initially K_1 is large and $K_2, K_3 \geqslant 0$.

For all α with $0 \leqslant \alpha < 1$, there is only one critical fixed point $\boldsymbol{K}^c$ with one relevant and two irrelevant scaling fields. This is to be expected since all models given by (6) with positive initial coupling constants should belong to the same universality class (Burkhardt 1979). The scaling index $y_t(\alpha)$ corresponding to the relevant scaling field as a function of α again has a unique maximum. The maximum occurs at $\alpha = \alpha^* = 0.42$ and $y_t(\alpha^*) = 0.887$. For $\alpha = \alpha^*$, the fixed point coordinates are $\boldsymbol{K}^c = (0.312, 0.095, -0.023)$. The physical trajectory $\boldsymbol{K} = (K_1, 0, 0)$ corresponding to the usual Ising model hits the critical surface at $K_1 = 0.385$ compared with the exact value 0.441.

The bond moving transformation just described implies a lower bound to the free energy for all α just as in the triangular case. Thus, we may apply the Kadanoff criterion (5) with $K^c(\alpha)$ replaced by $\boldsymbol{K}^c(\alpha)$. This leads to $\alpha^{**} = 0.62$ which is again quite different from α^*. However, for the Ising model (6) we can modify the bond moving

scheme in such a way that $\alpha^{**} \simeq \alpha^{*}$. One way to achieve this is to move the interaction terms of the form $K_3\sigma_1\sigma_2\sigma_3\sigma_4$ in (6) in a different fashion as indicated in figure 3. In this figure, every circle now represents four interaction terms of the above form. Summing over all spins indicated by white vertices in figure 3, we obtain new recursion relations of

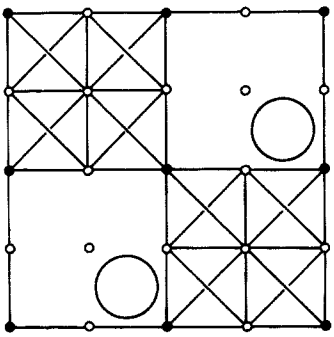

Figure 3. Modified bond moving step for Ising model with checkerboard interactions.

the generic form (3) and (7). As before, the MKRG scheme is recovered for $\alpha = 0$. In the low-temperature limit, these new recursion relations reduce to

$$K_1' = (2-\alpha)K_1 - \tfrac{1}{2}K_3, \tag{9a}$$

$$K_2' = \tfrac{1}{4}\ln 2, \tag{9b}$$

$$K_3' = \begin{cases} -\alpha K_1 + \tfrac{3}{2}K_3, & 0 \leqslant \alpha < \tfrac{1}{2}, \\ -(2-\alpha)K_1 + \tfrac{1}{2}K_3, & \tfrac{1}{2} \leqslant \alpha < 1, \end{cases} \tag{9c}$$

and

$$\delta f_{\mathrm{B}}(\boldsymbol{K}) = 2(2+\alpha)K_1 + K_3. \tag{9d}$$

This asymptotic behaviour ensures that we approach the ferromagnetic low-temperature fixed point such that the approximate bulk free energy becomes asymptotically exact for all α. For this changed bond moving scheme, we find the same phase diagram as before. Our criterion $\partial y_t(\alpha)/\partial\alpha|_{\alpha^*} = 0$ leads to $\alpha^* = 0.58$ while the Kadanoff criterion again yields $\alpha^{**} = 0.62$. The corresponding relevant scaling indices are $y_t(\alpha^*) = 0.972$ and $y_t(\alpha^{**}) = 0.969$ which are only 3% off the exact value.

The Ising model as described by the sub-Hamiltonian (6) has been considered previously by Burkhardt (1979). He combined a duality-decimation transformation with a bond moving transformation. The scaling index obtained in this way is as accurate as the value obtained by our second bond moving scheme (indicated in figure 3). The duality-decimation transformation has the nice feature that it contains no free parameters. However, when applied to model (6) this transformation leads to an incorrect phase diagram since there are several fixed points. In contrast, since there is only one critical fixed point with only one relevant perturbation our bond moving scheme yields the expected phase diagram.

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